

Conformality Lost

M. Stephanov

U. of Illinois at Chicago

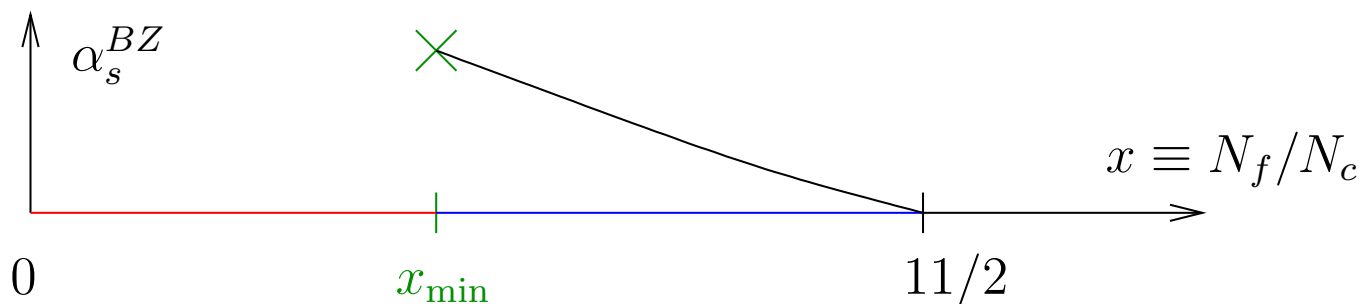
with D. Kaplan, J-W. Lee, D. Son (*INT, Seattle*)

Conformality

- In a conformal theory nothing is special about any distance scale.
- This is nontrivial for an *interacting* QFT, because couplings generally *run*.
- $\Lambda \frac{\partial g}{\partial \Lambda} = \beta(g)$. Thus $\beta(g_c) = 0$ in a conformal point $g = g_c$.
- Well-known example is QCD at the Banks-Zaks conformal point:

$$2\pi\beta(\alpha_s) = \beta_0\alpha_s^2 + \beta_1\alpha_s^3 + \dots = 0$$

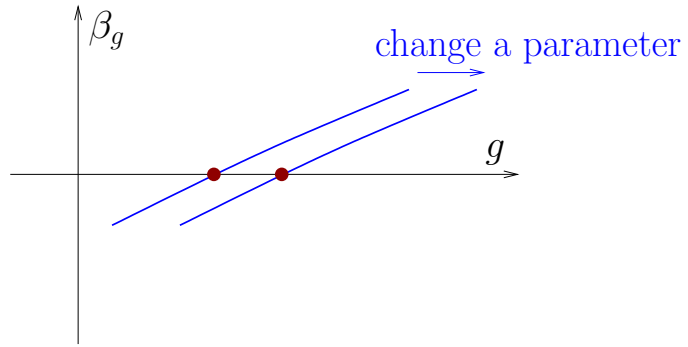
has a nontrivial solution $\alpha_s^{BZ} \approx -\beta_0/\beta_1$
 in perturbative domain, when $\beta_0 = (11N_c - 2N_f)/3$ is small.
 For $N_c \rightarrow \infty$ QCD, $\alpha_s^{BZ} \rightarrow 0$, when $x \equiv N_f/N_c \rightarrow 11/2$.



conformality lost conformal window

Conformality loss

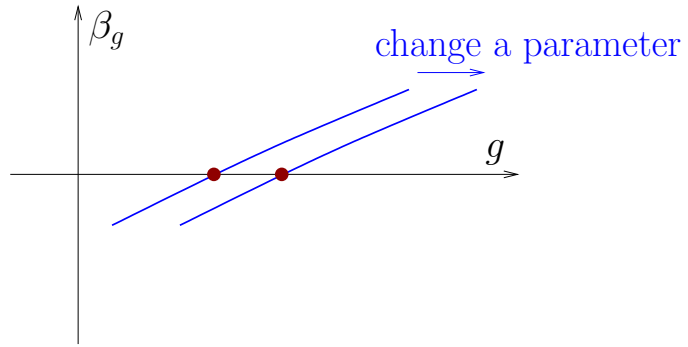
Change a parameter and g_c only shifts:



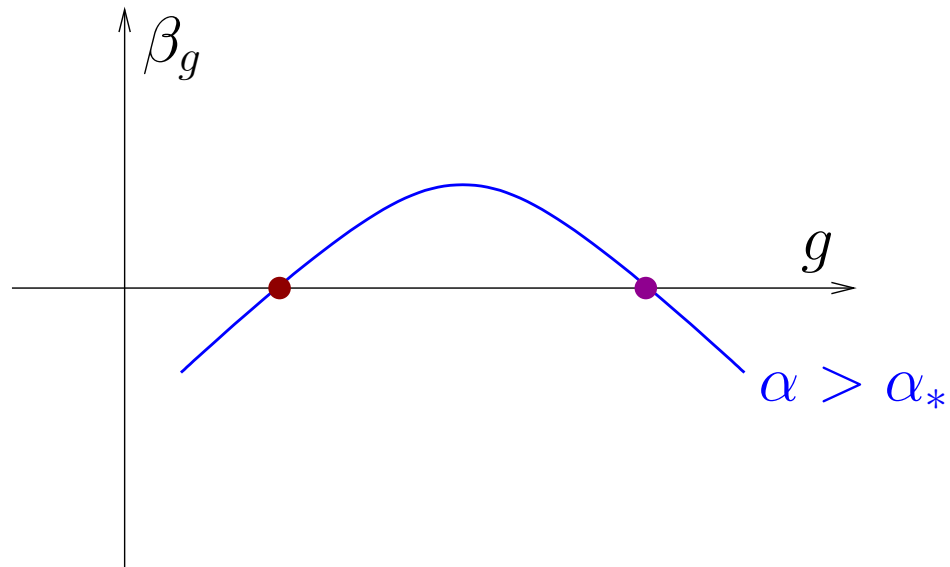
unless it collides with another fixed point:

Conformality loss

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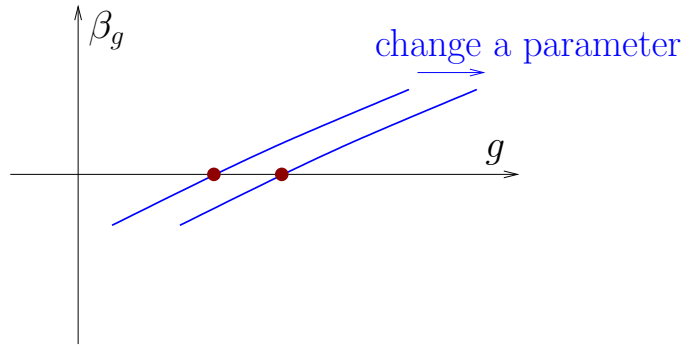


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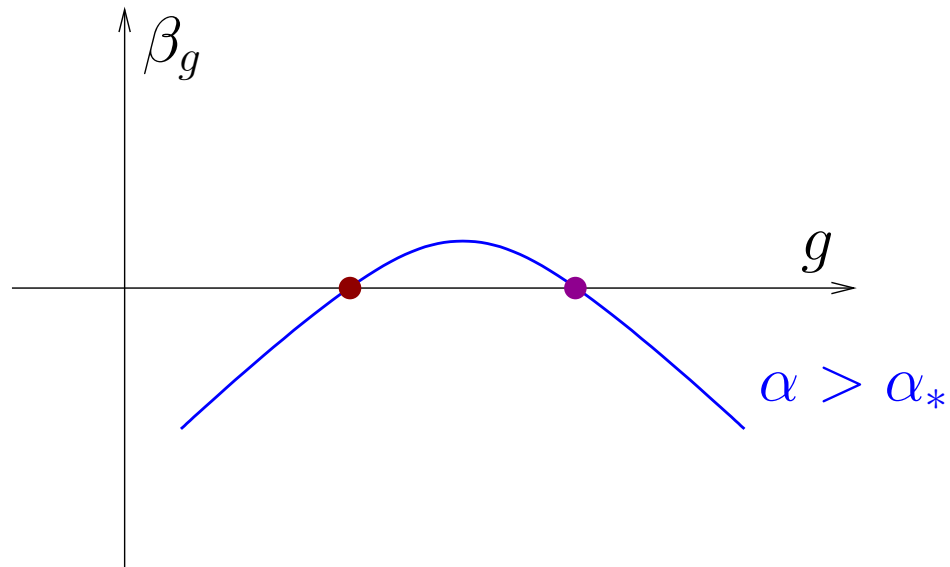


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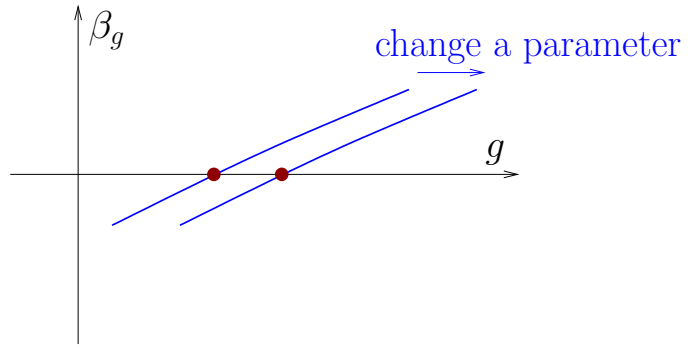


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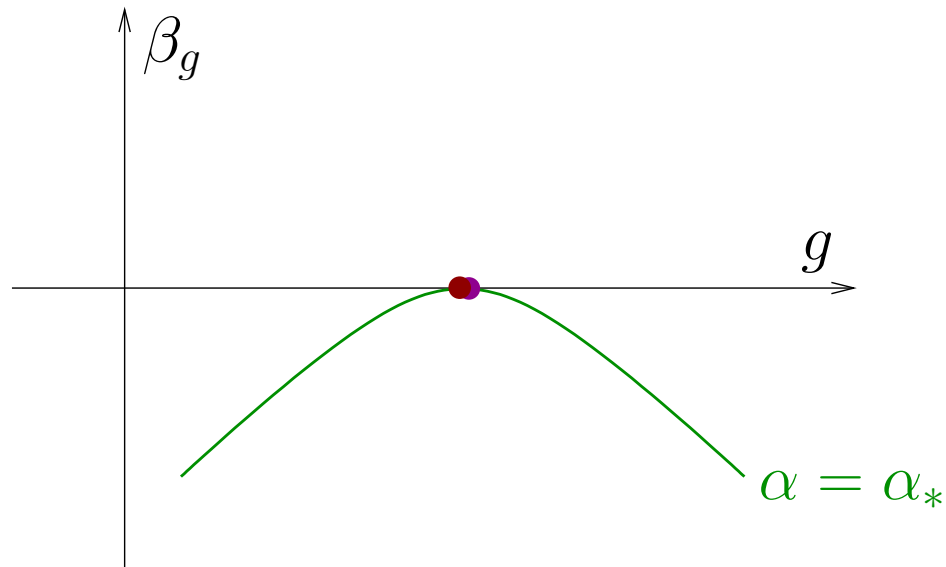


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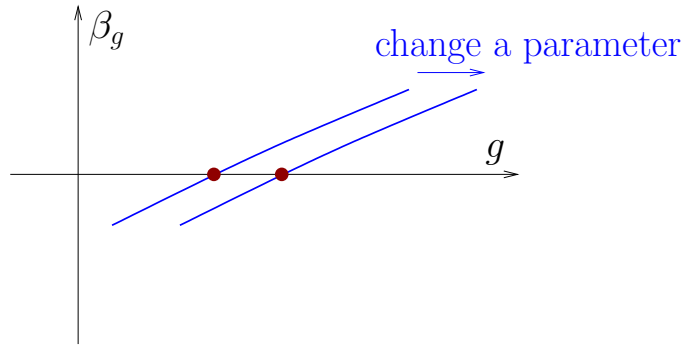


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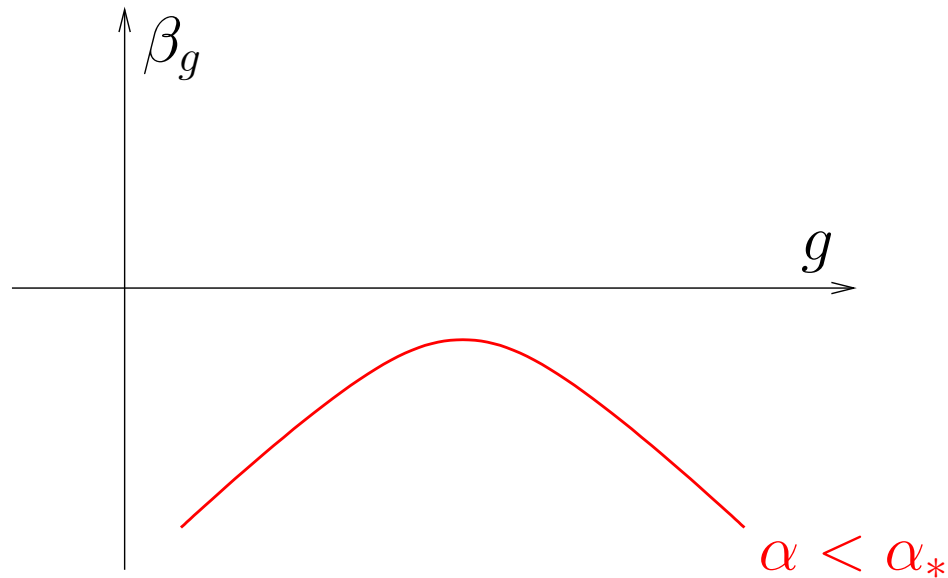


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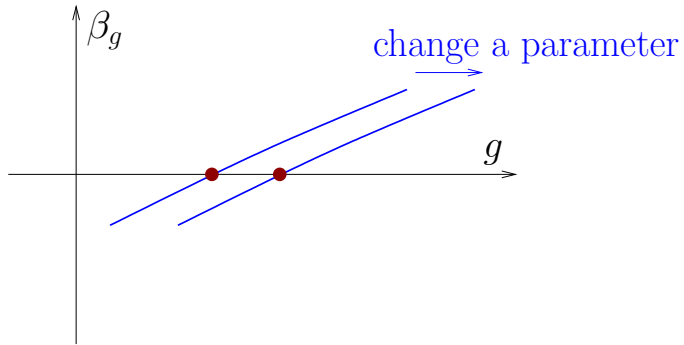


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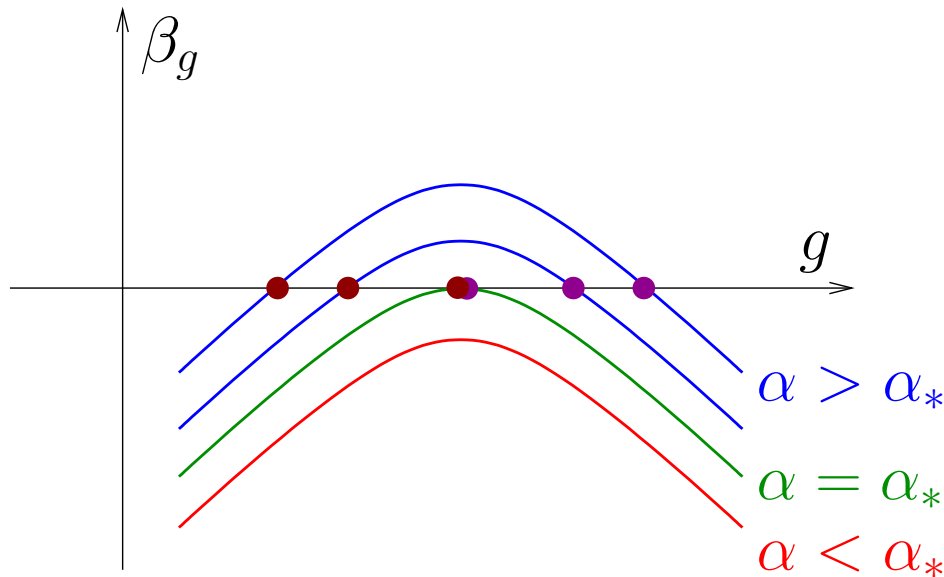


Conformality loss

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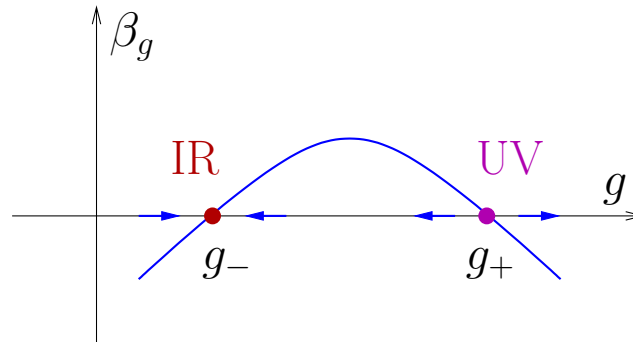
unless it collides with another fixed point:



● Let us explore this mechanism of fixed point “annihilation”.

RG flows and scale generation

● IR-UV pair:



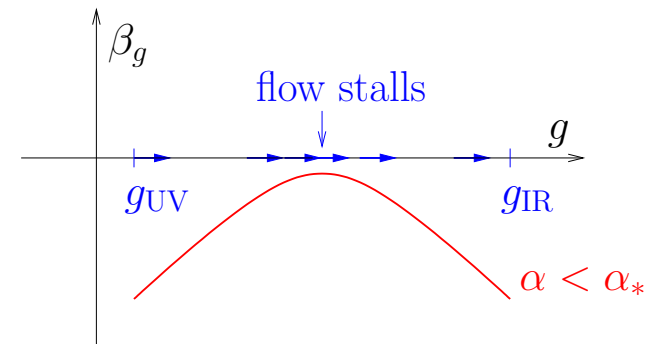
● Model:

$$\beta_g = (\alpha - \alpha_*) - (g - g_*)^2 = (g - g_-)(g - g_+).$$

$$g_{\pm} = g_* \pm \sqrt{\alpha - \alpha_*}.$$

We can integrate the RG equation:

$$\frac{\Lambda_{\text{IR}}}{\Lambda_{\text{UV}}} = \exp \int_{g_{\text{UV}}}^{g_{\text{IR}}} \frac{dg}{\beta_g} \xrightarrow{\alpha \rightarrow \alpha_*} \exp \left(-\frac{\pi}{\sqrt{\alpha_* - \alpha}} \right)$$



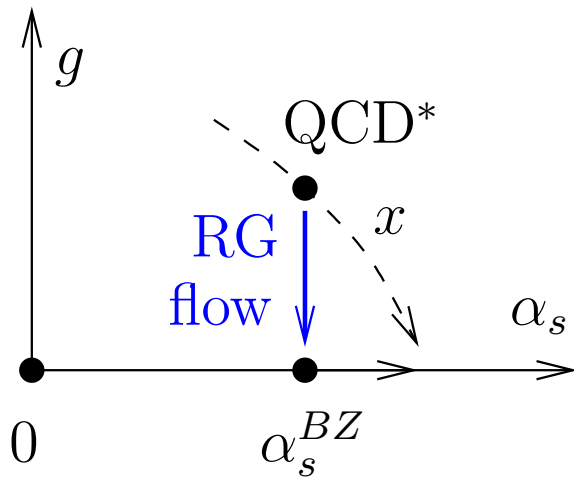
This is the well-known BKT scaling!

● The slope $\beta'(g_-) = 2\sqrt{\alpha - \alpha_*}$ is the scaling dimension of the deformation $g - g_-$.

Near the critical point α_* this deformation (irrelevant for IR point) becomes marginal.

Examples

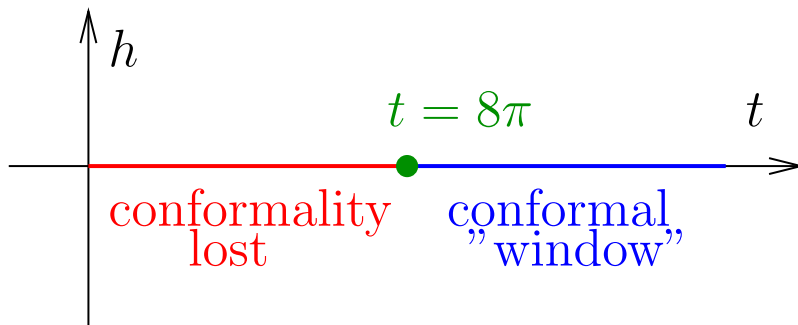
- XY model.
- QM with $V = 1/r^2$.
- Holographic model.
- QCD*?



XY model

- A two-component classical spin model, which can be described by a 2d field theory:

$$\mathcal{L} = \frac{1}{t} \left[\frac{1}{2} (\partial_\mu \theta)^2 - h \cos \theta \right]$$



- In 2d continuous symmetry cannot be spont. broken, but transitions can and do occur.

Physics: equivalent to Coulomb gas of vortices, at dual $T = 1/t$.

$E \sim \log R$ and vortices are bound in zero-vorticity pairs for small T .

But $S \sim \log R$. For large enough T , S wins over E/T and vortices inbind, screening the Coulomb potential: $\xi < \infty$.

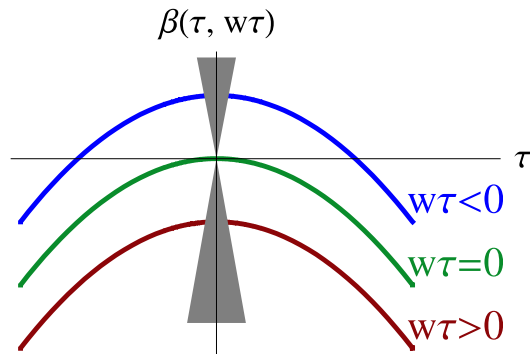
XY model: RG treatment

● In terms of $u = 1 - t/(8\pi)$ and $v = h/\Lambda^2$:

$$\beta_u = -2v^2, \quad \beta_v = -2uv$$

or using $\tau = u + v$, $2w = v - u$:

$$\beta_\tau = -2w\tau - \tau^2, \quad w\tau = \text{RG invariant}$$



● With $\alpha - \alpha_* = -2w\tau$, $\tau = g - g_*$, same as in toy RG model.


(Caveat: gray region is non-perturbative in u, v .)

●


$$\xi = \Lambda \exp \left(\frac{\pi}{2\sqrt{2w\tau}} \right) \sim e^{\text{const}/\sqrt{t_c - t}}$$

● Scaling dimension $[\cos \theta] = t/(4\pi) \rightarrow 2$
as $t \rightarrow 8\pi$, and h becomes relevant for $t < 8\pi$.

Quantum mechanics of $1/r^2$


$$i\frac{\partial\Psi}{\partial t} = \left[-\nabla^2 + \frac{\alpha}{r^2}\right]\Psi$$



is scale invariant (naively).



General solution for $E = 0$ (or any E at small r):

$$\Psi = c_- r^{\nu_-} + c_+ r^{\nu_+}, \quad \nu_{\pm} = -\sqrt{-\alpha_*} \pm \sqrt{\alpha - \alpha_*}, \quad \alpha_* = -\frac{(d-2)^2}{4}.$$

valid in the range $\alpha_* \leq \alpha \leq \alpha_* + 1$.

- 
- If c_- or c_+ are zero - the solution is scale invariant.
Otherwise c_+/c_- is a *dimensionful* parameter.
- 
- For $r \rightarrow \infty$ solution $\psi \rightarrow c_+ r^{\nu_+}$ — “IR fixed point.”

Quantum mechanics and RG

- To make sense of b.c. at $r = 0$, regularize:

$$V(r) = \begin{cases} \alpha/r^2, & r > r_0, \\ -g/r_0^2, & r < r_0, \end{cases}$$

Then

$$\frac{c_+}{c_-} = r_0^{(\nu_- - \nu_+)} \frac{\gamma + \nu_-}{\gamma + \nu_+}, \quad \gamma \equiv \left[\frac{\sqrt{g} J_{\frac{d}{2}}(\sqrt{g})}{J_{\frac{d-2}{2}}(\sqrt{g})} \right].$$

- The physics, i.e., c_+/c_- , is independent of r_0 if γ “runs”:

$$\beta_\gamma = -r_0 \frac{\partial \gamma}{\partial r_0} = -(\gamma + \nu_+)(\gamma + \nu_-) = (\alpha - \alpha_*) - (\gamma - \gamma_*)^2,$$

Same as in toy RG model. $\gamma = -\nu_-$ corresponds to $c_- = 0$ — IR fixed point, and $\gamma = -\nu_+$ — UV fixed point.

- For $\alpha < \alpha_*$ the ground state energy is

$$-E_0 = \frac{1}{r_0^2} \exp \left(-\frac{2\pi}{\sqrt{\alpha_* - \alpha}} + O(1) \right),$$

“Field-theory” treatment

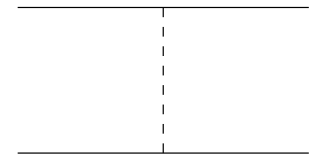
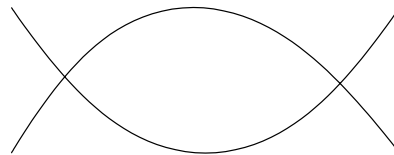
$$S = \int dt d^d \mathbf{x} \left(i\psi^\dagger \partial_t \psi - \frac{|\nabla \psi|^2}{2} + \pi \frac{g}{4} \psi^\dagger \psi^\dagger \psi \psi \right) \\ - \int dt d^d \mathbf{x} d^d \mathbf{y} \psi^\dagger(t, \mathbf{x}) \psi^\dagger(t, \mathbf{y}) \frac{\alpha}{|\mathbf{x} - \mathbf{y}|^2} \psi(t, \mathbf{y}) \psi(t, \mathbf{x}),$$

Feynman rules, $\epsilon = d - 2$:

Particle propagator: $\frac{i}{\omega - \mathbf{p}^2/2}$,

Contact vertex: $i\pi g \mu^{-\epsilon}$,

Static $1/r^2$ potential: $\frac{2\pi i \alpha}{\epsilon} \frac{1}{|\mathbf{q}|^\epsilon}$.



$$\beta_g = \epsilon g - \frac{g^2}{2} + 2\alpha = 2 \left(\alpha + \frac{\epsilon^2}{4} \right) - \frac{1}{2} (g - \epsilon)^2,$$

same as before, for small ϵ .

● One can also calculate the scaling dimension of $\psi\psi$:

$$[\psi\psi]_{\text{UV/IR}} = \frac{d+2}{2} \pm \sqrt{\alpha - \alpha_*}$$

Holographic dictionary (AdS/CFT)

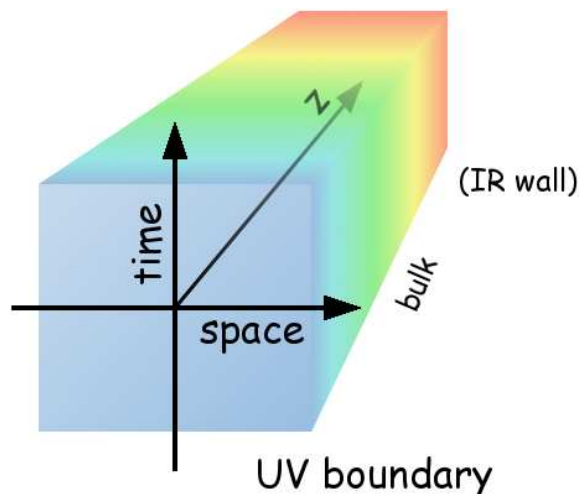
- Gauge theory in 4d defines a generating functional for Green's functions:

$$Z_4[\phi_0] = \int \mathcal{D}(\text{4d fields}) \exp\{iS + i \int_{x^4} \phi_0 \mathcal{O}\}$$

- Dual holographic theory lives in 5d and defines an effective action functional:

$$Z_5[\phi_0] = \int_{\phi(z \rightarrow 0) \rightarrow \phi_0} \mathcal{D}(\text{5d fields}) \exp\{iS_5\}$$

Duality means $Z_4 = Z_5$.



5d bulk metric:

$$ds^2 = z^{-2} (-dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu).$$

$$\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1).$$

(Note: $x^m \rightarrow \lambda x^m$).

Dimension of 4d operator and 5d mass

- Consider 5d action for a bulk scalar, dual to a scalar operator \mathcal{O} :

$$\mathcal{L}_5 = \frac{1}{2} \sqrt{g} (g^{mn} \partial_m \phi \partial_n \phi - m_{\text{bulk}}^2 \phi^2) \quad [g_{mn}] = -2, [\phi] = [m_{\text{bulk}}] = 0$$

$z \rightarrow 0$ (at fixed q , i.e., $qz \ll 1$) extremum satisfies

$$\partial_z (z^{-3} \partial_z \phi) - z^{-5} m_{\text{bulk}}^2 \phi = 0$$

$$\phi \sim z^{\Delta_\phi} \quad \text{with} \quad (\Delta_\phi - 4)\Delta_\phi - m_{\text{bulk}}^2 = 0.$$

- To make sense of the b.c., regulate at $z = \epsilon$:

$$\underline{\phi \epsilon^{-\Delta_\phi} = \phi_0} \quad - \quad \text{the source for } \mathcal{O}$$



$$[\phi] = 0 \quad \Rightarrow \quad [\phi_0] = +\Delta_\phi \quad ([x] = -1)$$

$$\text{Thus } [\mathcal{O}] = 4 - \Delta_\phi \equiv \Delta_{\mathcal{O}} \quad \text{and} \quad m_{\text{bulk}}^2 = \Delta_{\mathcal{O}}(\Delta_{\mathcal{O}} - 4)$$

Expectation value of an operator

- Find extremum of

$$S_{d+1} = \frac{1}{2} \int d^d x \sqrt{g} g^{mn} \partial_m \phi \partial_n \phi + \dots$$

with b.c. at $z = \epsilon \rightarrow 0$: $\phi \epsilon^{-\Delta_\phi} = \phi_0$. For small z :

$$\phi_{\text{sol}} = \alpha z^{\Delta_\phi} + \beta z^{\Delta_\mathcal{O}} \quad (\Delta_\phi + \Delta_\mathcal{O} = d).$$

- Calculate action (use e.o.m.):

$$S_{d+1} = \int_{x^d} \frac{\phi' \phi}{z^{d-1}} \Big|_{z=\epsilon} = \int_{x^d} \phi_0^2 \Delta \epsilon^{2\Delta-d} + (d - 2\Delta) \beta \phi_0$$

where $\Delta \equiv \Delta_\phi$.

- Use holographic correspondence $W_d = S_{d+1}$:

$$\langle \mathcal{O} \rangle = \frac{\delta W_d}{\delta \phi_0} = \frac{\delta S_{d+1}}{\delta \phi_0} = (2\Delta_\mathcal{O} - d) \beta + \text{contact terms}$$

I.e., $\alpha \sim \phi_0$ — source, $\beta \sim \langle \mathcal{O} \rangle$ — expectation value (response).

Pair of CFTs

- Equation of motion has *two* solutions. Near $z = 0$:

$$\phi(z) \rightarrow \alpha z^{\Delta_-} + \beta z^{\Delta_+}$$

where

$$\Delta_{\pm} = d/2 \pm \sqrt{m_{\text{bulk}}^2 + d^2/4}$$

- Since αz^{Δ_-} dominates as $z \rightarrow 0$, we have to set $\alpha = \phi_0$, thus $\Delta_{\phi_0} = \Delta_- < d/2$, and $\Delta_{\mathcal{O}} = \Delta_+ > d/2$.
- As observed by Breitenlohner-Freedman and Klebanov-Witten, for $-d^2/4 < m_{\text{bulk}}^2 < -d^2/4 + 1$, or $d/2 - 1 < \Delta_- < d/2$, there is an alternative CFT, with $\Delta_{\mathcal{O}} = \Delta_- < d/2$.

The alternative CFT, however, is not IR stable. Fine-tuning is necessary.

- There is a relevant deformation: \mathcal{O}^2 ($\Delta_{\mathcal{O}^2} < d$), which will “flow” to the original CFT.

Can see this in holography by adding $\frac{1}{2}c_0\mathcal{O}^2$ and tuning c_0 to obtain the alternative CFT (Witten, Gubser-Klebanov).

Alternative CFT

- Replace $\frac{1}{2}c_0\mathcal{O}^2$ by $\sigma\mathcal{O} + \sigma^2/2c_0$ and integral over σ .

Repeat the calculation of the extremum of the action, but now apply b.c.

$\phi = (\phi_0 + \sigma)\epsilon^\Delta$ at $z = \epsilon$.

- Calculate action (use e.o.m.):

$$S_{d+1} = \int_{x^d} \frac{\phi' \phi}{z^{d-1}} \Big|_{z=\epsilon} = \int_{x^d} (\phi_0 + \sigma)^2 \Delta \epsilon^{2\Delta-d} + (d - 2\Delta)\beta(\phi_0 + \sigma) + \frac{\sigma^2}{2c_0}$$

where $\Delta \equiv \Delta_-$.

- Integration over σ amounts to $\delta S/\delta\sigma = 0$. For a fine-tuned choice of c_0 (to cancel σ^2), this gives:

$$\beta = \phi_0 \frac{2\Delta}{(d - 2\Delta)\epsilon^{d-2\Delta}}$$

while α remains unconstrained by the b.c. at $z = \epsilon$.

I.e., α and β exchange roles.

Below BF bound

- At $m_{\text{bulk}}^2 = -d^2/4$, $\Delta_- = \Delta_+ = d/2$ and the two CFTs are the same.

What happens below, $m_{\text{bulk}}^2 < m_{\text{BF}}^2 \equiv -d^2/4$?

- Look at the e.o.m. again, write it as ($\psi = z^{\frac{d-1}{2}} \phi$)

$$-\psi'' + \frac{(4m_{\text{bulk}}^2 + d^2) - 1}{4z^2} \psi = q^2 \psi$$

Same as QM in 2d with $V(r) = (4m_{\text{bulk}}^2 + d^2)/r^2$, $E = q^2$.

When $(4m_{\text{bulk}}^2 + d^2) < 0$, there is a bound state

with energy $q^2 \sim -\epsilon^{-2} \exp(-2\pi/\sqrt{m_{\text{BF}}^2 - m_{\text{bulk}}^2})$ — tachyon.

BKT scaling again!

- Presumably, (in a more complete gravity dual?) tachyon instability would be cured if an IR wall (cutoff) dynamically develops at

$$z_{\text{IR}} \sim \epsilon \exp(\pi/\sqrt{m_{\text{BF}}^2 - m_{\text{bulk}}^2}).$$

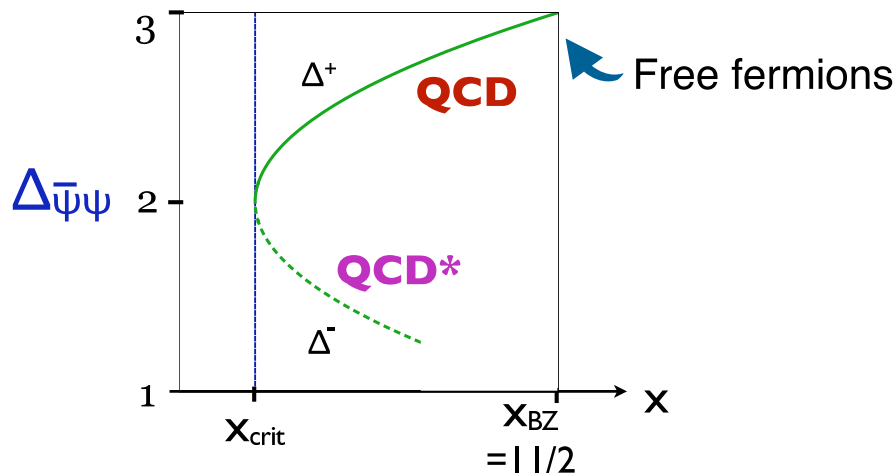
In search of QCD*

- What could this all tell us about QCD* (if it exists)?

- Expect a pair of scalar operators with $\Delta_- + \Delta_+ = 4$.

A natural choice is $\bar{\psi}\psi$, which at $x = 11/2$ has $\Delta_+ = 3$.

Thus in QCD* expect an operator with $\Delta_- = 1$. Free scalar?



- For example, we can try:

$$\mathcal{L}_{\text{model A}} = \mathcal{L}_{\text{QCD}} + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{y}{\sqrt{2}}\bar{\psi}\psi\phi - \frac{\lambda}{24}\phi^4.$$

Model A (QCD* prototype)

$$a_s = \frac{g^2 N_c}{(4\pi)^2}, \quad a_y = \frac{y^2 N_c N_f}{(4\pi)^2}, \quad \hat{\lambda} = \frac{\lambda N_c N_f}{(4\pi)^2}.$$

$$\beta_{a_s} = -\frac{2}{3}[(11 - 2x)a_s^2 + (34 - 13x)a_s^3],$$

$$\beta_{a_y} = -6a_s a_y + 2a_y^2,$$

$$\beta_{\hat{\lambda}} = -12a_y^2 + 4a_y \hat{\lambda}$$

$$a_{y*} = 3a_{s*}, \quad \hat{\lambda} = 3a_{y*} = 9a_{s*}.$$

➡ Thus, model A has a perturbative fixed point.

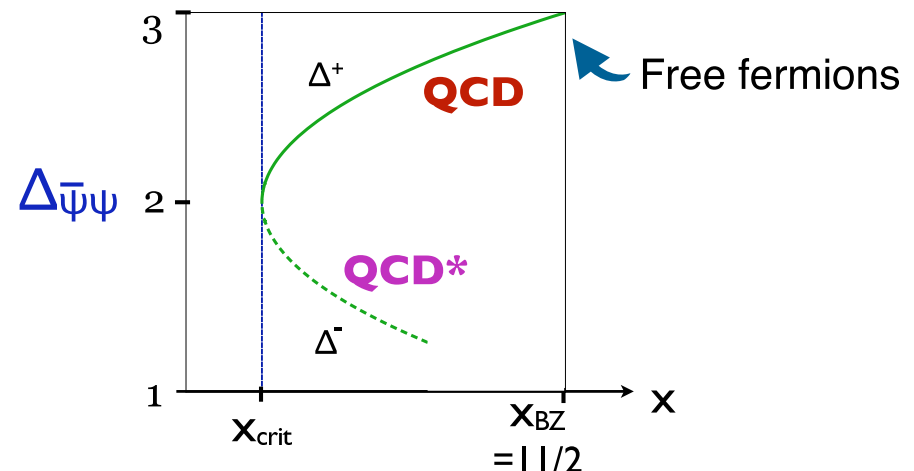
Moreover:

$$\Delta_+ = \Delta[\bar{\psi}\psi]_{\text{BZ}} = 3 - 3a_{s*},$$

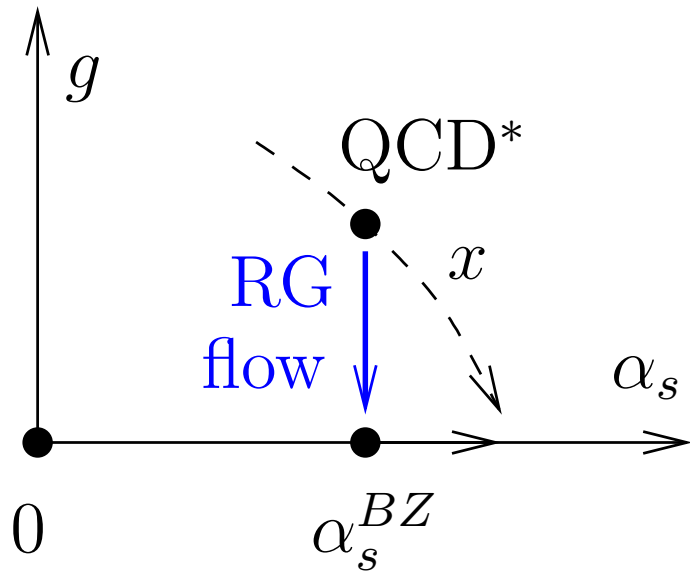
$$\Delta_- = \Delta[\phi]_{\text{model A}} = 1 + a_{y*}.$$

and, as expected,

$$\Delta_+ + \Delta_- = 4,$$



The relevance of the scalar mass



The relevant deformation for model A (QCD^* prototype) is, naturally, $m^2\phi^2$ — scalar mass.

Beyond model A

- Model A does not have the full global symmetry of QCD $SU(N_f) \times SU(N_f)$.

A generalization of this model ($2N_f^2$ scalars) that does have the full symmetry does not have a perturbative fixed point. Perhaps, it does near x_{crit} ?

- Some light can be shed using $2M^2$ scalars, with $M = N_f/k$, which has smaller symmetry, $SU(M) \times SU(M) \times SU(k)$.

- There is a perturbative fixed point for any $k > 1$, but not for $k = 1$.

- Interestingly, the operator dimensions sum up to (for $k \gg 1$, $x \rightarrow 11/2$)

$$\Delta_+ + \Delta_- = 4 + \frac{88}{625} \frac{n_\phi}{N_f^2} (11 - 2x),$$

where $n_\phi = 2M^2$.

- In holography, this can be understood as “Casimir effect”. Change of the b.c. changes the curvature radius R of the AdS metric. Thus

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m_{\text{bulk}}^2 R_\pm^2}$$

if $R_+/R_- = 1 + O(n_\phi/N^2)$, then $\Delta_+ + \Delta_- = 4 + O(n_\phi/N_c^2)$.

Summary

- Fixed point annihilation is a ubiquitous mechanism of conformality loss.
- Very natural phenomenon in AdS/CFT holography.
- Implications for QCD:

- Leads to BKT scaling below x_{\min} (below BF bound in holography).

In the context of QCD, first found by Miransky *et al*, using SD approach. Although SD approximation is uncontrollable, the scaling is generic, as our RG treatment shows.

Important for lattice studies determining x_{\min} .

- Predicts existence of QCD* — conformal theory with one unstable (RG relevant) direction, which flows into BZ fixed point.
- We could not find QCD* for $x \approx 11/2$. Perhaps, it exists only near x_{\min} ?

